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Empirical Topology: Topologies from Partially Ordered Sets

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A generalization of point-set topology is constructed in terms of arbitrary partially ordered sets. The possible usefulness of this construction in quantum theory, and specifically quantum gravitation, is discussed.

1. INTRODUCTION

Starting with the notion of a filter in a partially ordered set (poset), we define topological structures in such a way that each topological structure on a poset defines an ordinary point-set topology on a subset of the maximal filters of the poset. We then show that if the initial poset is the power set of some set S, ordered by inclusion, a subclass of our topological structures defines all possible point-set topologies on S. The advantages of the more general approach for physical purposes becomes evident when the poset Q we start with is the set of empirical questions on a physical system, ordered by states, since then topological properties of any structure resulting from Q can be formulated purely in terms of the quantum structure of Q. In our last section we discuss this aspect of the construction in the context of a possible approach to quantum gravity.

Throughout this paper, D, T, and L designate definitions, theorems and lemmas, respectively.

2. GENERAL T STRUCTURES

D1. The term *poset* will mean an arbitrary partially ordered set containing more than one element. Whenever a statement refers to 0 (the least element) or 1 (the greatest element), the reference is to be understood

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nonrestrictively, i.e., the statement applies to arbitrary posets and the reference to 0 or 1 applies in the special case where such an element exists. For $\{Q, \leqslant\}$ a poset and $R \subset Q$, define

$$R^{\vee} = \{t \in Q \colon \forall r \in R, r \leqslant t\}$$

$$R^{\wedge} = \{t \in Q \colon t \neq 0 \text{ and } t \leqslant r, \forall r \in R\}$$

A nonempty subset $F \subset Q$ will be called a *filter* in Q if

(F1) $q \in F \Rightarrow \{q\}^{\vee} \subset F$

(F2) $\{q,r\} \subset F \Rightarrow \{q,r\}^{\wedge} \cap F \neq \varnothing$

Let $\mathfrak{F}_Q$ be the set of filters in Q, partially ordered by set inclusion, and let $\Omega_Q \subset \mathfrak{F}_Q$ be the set of *maximal filters* in Q, i.e.,

$$\Omega_Q = \{F \in \mathfrak{F}_Q \colon \forall G \in \mathfrak{F}_Q, F \subset G \Rightarrow F = G\}$$

We will designate elements of Ω_Q by lower case Greek letters when we need to distinguish them from arbitrary filters (to be designated still by upper case Roman letters).

L1. For Q a poset, $\mathfrak{F}_Q \neq \varnothing, \Omega_Q \neq \varnothing$, and $[1 \in F, 0 \notin F], \forall F \in \mathfrak{F}_Q$.

Proof. First, (F1)$\Rightarrow 1 \in F, \forall F \in \mathfrak{F}_Q$, and since $\{q,0\}^{\wedge} = \varnothing, \forall q \in Q$, (F2)$\Rightarrow 0 \notin F, \forall F \in \mathfrak{F}_Q$. Then, it is trivial that $q \neq 0 \Rightarrow \{q\}^{\vee} \in \mathfrak{F}_Q$. Hence, $\mathfrak{F}_Q \neq \varnothing$. Finally, it is easy to check that the union of any chain of filters is itself a filter, and so $\mathfrak{F}_Q$ contains an upper bound for any of its chains; therefore, by Zorn's lemma, $\Omega_Q \neq \varnothing$. ■

D2. For Q a poset, $q \in Q$ and $\mu \in \Omega_Q$, we will write $\mu \varepsilon q$ if $q \in \mu$. A *topological structure* (T structure) will mean a triplet $\{Q, \Delta, T\}$, where Q is a poset, Δ is a subset of Ω_Q, and T is a mapping

$$T \colon \Delta \to \mathfrak{F}_Q \colon \mu \to T_\mu$$

such that, $\forall \mu \in \Delta$,

(τ1) $T_\mu \subset \mu$

(τ2) $\forall q \in T_\mu, \exists r \in T_\mu \colon r \leqslant q$ and $r \in \bigcap_{\nu \varepsilon r} T_\nu$

Q will be called the *poset*, Δ the *domain*, and T the *topology* of the T structure. For brevity, a T structure will often be denoted by the topology T of the structure. An element $q \in Q$ will be said to be a *neighborhood* in T of a maximal filter $\mu \in \Delta$ if $q \in T_\mu$, and T_μ will be called the *filter of neighborhoods* of μ in T. An element $q \in Q$ will be said to be *open* in T if $q \in \cap_{\mu \varepsilon q} T_\mu$, i.e., if q is a neighborhood of all the maximal filters in Δ to which q belongs.

Finally, for Q a poset define

$$A_Q = \{q \in Q \colon \{q\}^{\wedge} = \{q\}\}$$

A_Q is called the set of atoms of Q, and Q is said to be atomic if $\forall q \in Q, \{q\}^\wedge \cap A_Q \neq \varnothing$.

L2. If Q is an atomic poset, the correspondence μ defined by

$$a \to \mu_a = \{a\}^\vee, \qquad a \in A_Q$$

is an injection from A_Q into Ω_Q.

Proof. We have already noted in the proof of L1 that $\{a\}^\vee$ is a filter. To see that it is maximal, assume $\{a\}^\vee \subset F \in \mathfrak{F}_Q$. Then, by (F2), $[q \in F] \Rightarrow [\{q,a\}^\wedge \neq \varnothing] \Rightarrow [\{q,a\}^\wedge \subset \{a\}^\wedge = \{a\}] \Rightarrow [q \in \{a\}^\vee]$. Hence, $\forall F \in \mathfrak{F}_Q, \{a\}^\vee \subset F \Rightarrow F \subset \{a\}^\vee$, and so $\{a\}^\vee \in \Omega_Q$, and the correspondence $a \to \mu_a = \{a\}^\vee$ is a mapping μ: $A_Q \to \Omega_Q$. That μ is one-to-one is seen from $a, b \in A_Q, [a \neq b] \Rightarrow [\{a,b\}^\wedge = \varnothing] \Rightarrow [a \notin \{b\}^\vee] \Rightarrow [\{a\}^\vee \neq \{b\}^\vee]$.

D3. A T structure $\{Q, \Delta, T\}$ will be said to be *atomic* if Q is atomic and Δ is a subset of $\mu(A_Q)$, where μ is the mapping defined by L2. Since μ is one-to-one, we can (and will) in this case identify Δ with the set of atoms $D = \mu^{-1}(\Delta)$, and so write an atomic T structure as $\{Q, D, T\}$ with

$$T\colon D \to \mathfrak{F}_Q\colon a \to T_a = T_{\mu_a}$$

A particularly important example of an atomic T structure is given by any standard point-set topological structure (Treves, 1967), called simply a *topological space* or TS, and specified by requiring that the poset Q be the power set $\mathcal{P}_S$ of some nonempty set S, equipped with the partial ordering of set inclusion (obviously $\mathcal{P}_S$ is then an atomic poset), and setting $D = A_Q = \{\{x\}\colon x \in S\}$. Since in this case D has a natural identification with S itself, it is customary to denote a TS simply by $\{S, T\}$, with $Q = \mathcal{P}_S$ understood, and the topology T considered as a mapping

$$T\colon S \to \mathfrak{F}_{\mathcal{P}_S}\colon x \to T_x$$

3. TOPOLOGICAL SPACES FROM T STRUCTURES

L3. Given a T structure $\{Q, \Delta, T\}$, if, for each $q \in \cup_{\mu \in \Delta} T_\mu$, we define

$$N_q = \{\mu \in \Delta\colon \mu \varepsilon q\}$$

then, for each $\mu \in \Delta$, the family of subsets

$$T_\mu^* = \{N \subset \Delta\colon \exists q \in T_\mu, N_q \subset N\}$$

is a filter in the poset $\{\mathcal{P}_\Delta, \subset\}$.

Proof. T^*_μ is nonempty, since $T_\mu \neq \varnothing \Rightarrow [\exists q \in T_\mu\colon N_q \neq \varnothing] \Rightarrow N_q \in T^*_\mu$, and (F1) is obviously satisfied, from the definition above. To see that (F2) is verified we have the following chain of implications: $[N \in T^*_\mu$ and $M \in T^*_\mu] \Rightarrow [\exists q, r \in T_\mu\colon N_q \subset N$ and $N_r \subset M] \Rightarrow [\{q,r\}^\wedge \cap T_\mu \neq \varnothing] \Rightarrow [\exists t \in T_\mu\colon t \neq 0,\ t \leqslant q$ and $t \leqslant r] \Rightarrow [N_t \in T^*_\mu]$, where $N_t = \{\nu \in \Delta\colon \nu \varepsilon t\}$. But $\nu \varepsilon t \Rightarrow t \in \nu \Rightarrow [\{q,r\} \subset \nu, \forall \nu \in N_t] \Rightarrow [\forall \nu \in N_t, \nu \in N_q$ and $\nu \in N_r] \Rightarrow [N_t \subset N_q \cap N_r] \Rightarrow [N_t \subset N \cap M] \Rightarrow [N \cap M \in T^*_\mu]$, by (F1). Hence T^*_μ is a filter of subsets in $\mathcal{P}_\Delta$. ■

D4. For $\{Q, \Delta, T\}$ a T structure, and $T^*_\mu, \mu \in \Delta$, the filter of subsets in Δ given in L3, let T^* be the mapping

$$T^*\colon \Delta \to \mathfrak{F}_{\mathcal{P}_\Delta}\colon \mu \to T^*_\mu$$

T1. For $\{Q, \Delta, T\}$ a T structure, $\{\Delta, T^*\}$ is a topological space (i.e., T^* is a point set topology on Δ).

Proof. From D3, what we need to show is that $\{\mathcal{P}_\Delta, \Delta^*, T^*\}$ is a T structure, where Δ^* is the set of maximal filters in $\mathcal{P}_\Delta$ generated by the atoms $\{\mu\} \in \mathcal{P}_\Delta$, i.e., $\Delta^* = \{\mu^*\colon \mu \in \Delta\}$, $\mu^* = \{N \subset \Delta\colon \mu \in N\}$, and $T^*_{\mu^*} = T^*_\mu, \mu \in \Delta$. But $N \in T^*_\mu \Rightarrow [\exists q \in T_\mu\colon N_q \subset N]$, and, from ($\tau$1), $q \in T_\mu \Rightarrow q \in \mu \Rightarrow \mu \varepsilon q \Rightarrow \mu \in N_q \Rightarrow \mu \in N \Rightarrow N \in \mu^*$. Hence, $\forall \mu^* \in \Delta^*, T^*_{\mu^*} \subset \mu^*$, and ($\tau$1) is verified for $\{\mathcal{P}_\Delta, \Delta^*, T^*\}$. To see that ($\tau$2) is also satisfied, we have, from the definition in L3, $r \leqslant q \Rightarrow N_r \subset N_q$, since $r \leqslant q$ implies $q \in \mu$ if $r \in \mu$, and from (τ2) applied to $\{Q, \Delta, T\}$ we have $\forall q \in T_\mu$, $\exists r \in T_\mu\colon r \leqslant q$ and $r \in \cap_{\nu \varepsilon r} T_\nu$. But $r \in \cap_{\nu \varepsilon r} T_\nu \Rightarrow [N_r \in T^*_\nu, \forall \nu \varepsilon r] \Rightarrow [N_r \in \cap_{\nu \varepsilon r} T^*_\nu] \Rightarrow [N_r \in \cap_{\nu \in N_r} T^*_\nu]$. Hence, $N \in T^*_{\mu^*} \Rightarrow [\exists q \in T_\mu\colon N_q \subset N] \Rightarrow [\exists r \in T_\mu\colon N_r \subset N$ and $N_r \in \cap_{\nu^* \in N_r} T^*_{\nu^*}]$, and so ($\tau$2) is established for $\{\mathcal{P}_\Delta, \Delta^*, T^*\}$, which implies that $\{\Delta, T^*]$ is a topological space in the ordinary meaning of point set topology. ■

4. POSSIBLE RELEVANCE FOR QUANTUM GRAVITY

If in a theory of quantum gravity we wish to preserve at least the spirit of Einstein's insight that gravitation should be considered, not as an additional fundamental force, but as an effect (curvature) produced by the presence of mass-energy-momentum in the differentiable manifold structure of classical space-time, then it becomes reasonable to postulate that space-time itself with its macroscopic differentiable topological structure, including gravitation as curvature, should result as a by-product of the other fundamental quantum interactions needed to explain physical reality. This viewpoint does not preclude the possibility that an additional fundamental field quantum may be needed (and it might well resemble the graviton of many current approaches), but it does not require the *a priori* introduction of such a particle. It does, however, go counter to any viewpoint that would make the classical differentiable space-time structure a necessary foundation for doing quantum field theory at all. Possible evidence favoring the viewpoint outlined here might be the facts that gravitation seems to be macroscopic (it seems so far to play no role in fundamental interactions) and that current theories, introducing classical differentiable space-time as a necessary background, run into divergence problems—general relativity in the high density limit with black holes and quantum field theory with its need for renormalization as soon as any interactions are introduced.

In any case, and certainly with the viewpoint proposed above, if there is to be a proper wedding of gravitation and quantum theory we should have a means of formulating topological properties resulting from various models in purely quantum terms. The general topological structures defined above provide this means. We hope to investigate in future what a generalization of differentiable manifolds and distribution theory might look like in terms of T structures.

APPENDIX

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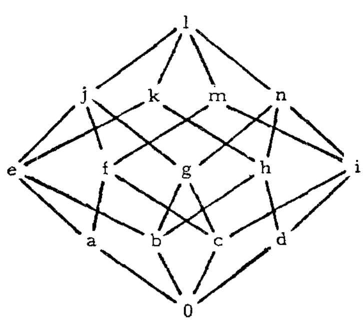
The following example, based on a small finite poset, illustrates a general T structure and the point-set topological space it generates. Since finite posets are atomic, we use the notation adapted for atomic T structures. The diagrams represent partial ordering by joining lines, the greater element having higher altitude.

Example:

$$Q = \{1, 0, a, b, c, d, e, f, g, h, i, j, k, m, n\}$$



$\{Q, \leq\}$

Card $Q=15$ is odd. So Q is not equivalent to a power set. Therefore, any T structure on Q is not equivalent to a standard point-set topological space.
$\mu_a=\{a\}^\vee=\{a,e,f,j,k,m,l\}$
$\mu_b=\{b\}^\vee=\{b,e,g,h,j,k,n,l\}$
$\mu_c=\{c\}^\vee=\{c,f,g,i,j,m,n,l\}$
$\mu_d=\{d\}^\vee=\{d,h,i,k,m,n,l\}$
Let $\Delta=\{\mu_a,\mu_b,\mu_c,\mu_d\}, D=\mu^{-1}(\Delta)=A_Q=\{a,b,c,d\}$

$$T: D\to\mathfrak{F}_Q : a\to T_a=T_{\mu_a}$$

$$T_a=\{a\}^\vee,\qquad T_b=\{g,j,n,l\},\qquad T_c=\{f,j,m,l\},\qquad T_d=\{d\}^\vee$$

The set of open elements in T is

$$\mathcal{O}=\{1,0,a,d,f,j,m\}$$

e, g, h, i, k, and n are neighborhoods, but are not open. Note that

$$0\in\bigcap_{\substack{\mu\varepsilon 0\\ \mu\in\Delta}} T_\mu=\bigcap_{\mu\in\varnothing} T_\mu=Q$$

since $0\notin\mu$ so $\mu\notin 0$ for all $\mu\in\Delta$, and the intersection over the empty subset of Δ is the universal set Q in which 0 is a member.

$$T^*: D\to\mathfrak{F}_{P_D} : a\to T_a^*=T_{\mu_a}^*$$

$$T_a^*=\{N\subset D:\exists q\,\varepsilon\, T_a, N_q\subset N\}$$

$$N_q=\{a\in D:\mu_a\,\varepsilon\, q\}, q\in\bigcup_{\mu\in\Delta} T_\mu$$

$$N_a=\{a\},\qquad N_f=\{a,c\},\qquad N_i=\{c,d\},\qquad N_m=\{a,c,d\}$$

$$N_d=\{d\},\qquad N_g=\{b,c\},\qquad N_j=\{a,b,c\},\qquad N_n=\{b,c,d\}$$

$$N_e=\{a,b\},\qquad N_h=\{b,d\},\qquad N_k=\{a,b,d\},\qquad N_l=\{a,b,c,d\}$$

$$T_a^*=\{\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},D\}$$

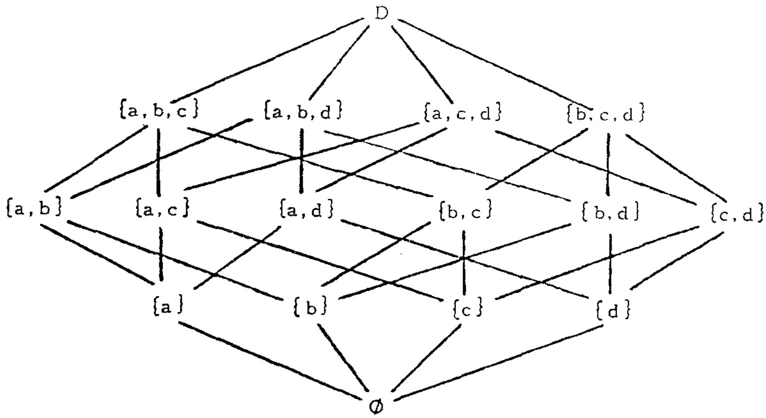
$$T_b^*=\{\{b,c\},\{b,c,d\},\{a,b,c\},D\}$$

$$T_c^*=\{\{a,c\},\{a,b,c\},\{a,c,d\},D\}$$

$$T_d^*=\{\{d\},\{a,d\},\{b,d\},\{c,d\},\{a,b,d\},\{a,c,d\},\{b,c,d\},D\}$$

The class of open sets in T^* is

$$O^*=\{\{a\},\{d\},\{a,c\},\{a,d\},\{a,b,c\},\{a,c,d\},D\}$$



$\{P_D \, , \; \subset\}$

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